Hamiltonian formalisms and symmetries of the Pais–Uhlenbeck oscillator

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Abstract

The study of the symmetry of Pais–Uhlenbeck oscillator initiated in Andrzejewski et al. (2014) [24] is continued with special emphasis put on the Hamiltonian formalism. The symmetry generators within the original Pais and Uhlenbeck Hamiltonian approach as well as the canonical transformation to the Ostrogradski Hamiltonian framework are derived. The resulting algebra of generators appears to be the central extension of the one obtained on the Lagrangian level; in particular, in the case of odd frequencies one obtains the centrally extended l-conformal Newton–Hooke algebra. In this important case the canonical transformation to an alternative Hamiltonian formalism (related to the free higher derivatives theory) is constructed. It is shown that all generators can be expressed in terms of the ones for the free theory and the result agrees with that obtained by the orbit method.

Introduction

The theories we are usually dealing with are Newtonian in the sense that the Lagrangian function depends on the first time derivatives only. There is, however, an important exception. It can happen that we are interested only in some selected degrees of freedom. By eliminating the remaining degrees one obtains what is called an effective theory. The elimination of a degree of freedom results in increasing the order of dynamical equations for remaining variables. Therefore, effective theories are described by Lagrangians containing higher order time derivatives [1]. Originally, these theories were proposed as a method for dealing with ultraviolet divergences [2]; this idea appeared to be quite successful in the case of gravity: the Einstein action supplied by the terms containing higher powers of curvature leads to a renormalizable theory [3]. Other examples of higher derivatives theories include the theory of the radiation reaction [4,5], the field theory on noncommutative spacetime [6,7], anyons [8,9] or string theories with the extrinsic curvature [10].

Of course, the appearance of terms with higher time derivatives leads to some problems. One of them is that the energy does not need to be bounded from below. To achieve a deeper insight into these problems and, possibly, to find a solution it is instructive to consider a quite simple, however nontrivial, higher derivatives theory. For example, it was shown in Ref. [11] (see also [12]) that the problem of the energy can be avoided (on the quantum level) in the case of the celebrated Pais–Uhlenbeck (PU) oscillator [13]. This model has been attracting considerable interest throughout the years (for the last few years, see, e.g., [11,12,14–24]). Recently, it has been shown (see [24]) that the properties of the PU oscillator, rather surprisingly, for some special values of frequencies change drastically and are related to nonrelativistic conformal symmetries. Namely, if the frequencies of oscillations are odd multiplicities of a basic one, i.e., they form an arithmetic sequences $\omega k = (2k - 1)\omega$, $\omega = 0$, for $k = 1,\dots, n$, then the maximal group of Noether symmetries of the PU Lagrangian is the l-conformal Newton–Hooke group with $l = 2n-1\ 2$ (for more informations about these groups see, e.g., [25–28] and the references therein). Otherwise, the symmetry group is simpler (there are no counterparts of dilatation and conformal generators (see the algebra (2.5)).

Much attention has been also paid to Hamiltonian formulations of the PU oscillator. There exists a few approaches to Hamiltonian formalism of the PU model: decomposition into the set of the independent harmonic oscillators proposed by Pais and Uhlenbeck in their original paper [13], Ostrogradski approach based on the

Ostrogradski method [29] of constructing Hamiltonian formalism for theories with higher time derivatives and the last one, applicable in the case of odd frequencies (mentioned above), which exhibits the l-conformal Newton–Hooke group structure of the model. Consequently, there arises a natural question about the relations between them as well as the realization of the symmetry on the Hamiltonian level? The aim of this work is to give the answer to this question.

The paper is organized as follows. After recalling, in Section 2, some informations concerning symmetry of the PU model on the Lagrangian level, we start with the harmonic decoupled approach. We find, on the Hamiltonian level, the form of generators (for both generic and odd frequencies) and we show that they, indeed, form the algebra which is central extension the one appearing on the Lagrangian level. Section 4 is devoted to the study of the relation between the above approach and the Ostrogradski one. Namely, we construct the canonical transformation which relates the Ostrogradski Hamiltonian to the one describing the decouple harmonic oscillator. This transformation enables us to find the remaining symmetry generators in terms of Ostrogradski variables. The next section is devoted to the case of odd frequencies where the additional natural approach can be constructed. In this framework the Hamiltonian is the sum of the one for the free higher derivatives theory and the conformal generator. We derive a canonical transformation which relates this new Hamiltonian to the one for the PU oscillator with odd frequencies. Moreover, we apply the method (see [30]) of constructing integrals of motion for the systems with symmetry to find all symmetry generators. Next, by direct calculations we show that they are related by the, above mentioned, canonical transformation to the ones of the PU model described in terms decoupled oscillators. We also express symmetry generators in terms of their counterparts in the free theory.

In concluding Section 6, we summarize our results and discuss possible further developments. Finally, Appendix A constitutes technical support for the mains results. We derive there some relations and identities which are crucial for our work.

Ostrogradski approach

Since the PU oscillator is an example of higher derivatives theory, it is natural to use the Hamiltonian formalism proposed by Ostrogradski [29]. To this end let us expand Lagrangian (2.1) in the sum of higher derivatives terms (here Q = x)

$$L = -\frac{1}{2}\vec{Q}\prod_{k=1}^{n}\left(\frac{d^2}{dt^2}+\omega_k^2\right)\vec{Q} = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k-1}\sigma_k\left(\vec{Q}^{(k)}\right)^2, \tag{4.1}$$

where

$$\sigma_k = \sum_{i_1<\ldots<i_{n-k}} \omega_{i_1}^2\cdots\omega_{i_{n-k}}^2, \quad k=0,\ldots,n, \quad \sigma_n = 1. \tag{4.2}$$

It can be shown (by standard reasoning) that the following identities hold

$$\sum_{k=1}^{n}\rho_k\omega_k^{2m} = 0, \quad m = 0,\ldots,n-2, \tag{4.3}$$

$$\sum_{k=1}^{n}\rho_k\omega_k^{2(n-1)} = (-1)^{n+1}, \tag{4.4}$$

$$\sum_{m=0}^{n}\sigma_m(-1)^m\sum_{k=1}^{n}\rho_k\omega_k^{2(r-n+m-1)} = 0, \quad r \geq n, \tag{4.5}$$

where ρ_k is given by Eq. (3.3). Now, we introduce the Ostrogradski variables

$$\begin{aligned} \vec{Q}_k &= \vec{Q}^{(k-1)}, \\ \vec{P}_k &= \sum_{j=0}^{n-k}\left(-\frac{d}{dt}\right)^j\frac{\partial L}{\partial\vec{Q}^{(k+j)}} = (-1)^{k-1}\sum_{j=k}^{n}\sigma_j\vec{Q}^{(2j-k)}, \end{aligned} \tag{4.6}$$

for $k = 1,\ldots,n$. Then the Ostrogradski Hamiltonian takes the form

$$H = \frac{(-1)^{n-1}}{2}\vec{P}_n^2 + \sum_{k=2}^{n}\vec{P}_{k-1}\vec{Q}_k - \frac{1}{2}\sum_{k=1}^{n}(-1)^k\sigma_{k-1}\vec{Q}_k^2. \tag{4.7}$$

By virtue of Eqs. (3.7) and (4.6), for $k = 1,\ldots,n$, we find

$$\begin{aligned} \vec{Q}_k &= (-1)^{\frac{k-1}{2}}\sum_{j=1}^{n}\sqrt{|\rho_j|}(-1)^{j-1}\omega_j^{k-1}\vec{x}_j, \quad k\text{-odd}; \\ \vec{Q}_k &= (-1)^{\frac{k}{2}-1}\sum_{j=1}^{n}\sqrt{|\rho_j|}\omega_j^{k-2}\vec{p}_j, \quad k\text{-even}; \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \vec{P}_k &= (-1)^{\frac{k}{2}-1}\sum_{i=1}^{n}(-1)^{i-1}\sqrt{|\rho_i|}\left(\sum_{j=k}^{n}\sigma_j(-1)^j\omega_i^{2j-k}\right)\vec{x}_i, \quad k\text{-even}; \\ \vec{P}_k &= (-1)^{\frac{k-1}{2}}\sum_{i=1}^{n}\sqrt{|\rho_i|}\left(\sum_{j=k}^{n}\sigma_j(-1)^j\omega_i^{2j-k-1}\right)\vec{p}_i, \quad k\text{-odd}. \end{aligned} \tag{4.9}$$

One can show that Eqs. (4.8) and (4.9) define a canonical transformation; to compute the Poisson brackets {Q k,Qj } and {Q k,P j } we use (4.3) and (4.3)–(4.5), respectively; computing {Pk,P j } is the most complicated one and involves considering two cases k − j ≶ 1 as well as applying Eqs. (4.3) and (4.5).

Next, let us note that the inverse transformation is of the form

$$\tilde{x}_i = \sum_{k=1}^{n}{}' (-1)^{\frac{k-1}{2}} \sum_{j=k}^{n} \sigma_j (-1)^j \omega_i^{2j-k-1} \sqrt{|\rho_k|}\,\tilde{Q}_k + \sum_{k=1}^{n}{}'' (-1)^{\frac{k}{2}} \sqrt{|\rho_k|}\,\omega_i^{k-2}\tilde{P}_k,$$

$$\tilde{p}_i = \sum_{k=1}^{n}{}'' (-1)^{\frac{k}{2}+i-1} \sum_{j=k}^{n} \sigma_j (-1)^j \omega_i^{2j-k} \sqrt{|\rho_k|}\,\tilde{Q}_k + \sum_{k=1}^{n}{}' (-1)^{\frac{k+1}{2}+i} \sqrt{|\rho_k|}\,\omega_i^{k-1}\tilde{P}_k. \quad (4.10)$$

No, we can try to find the symmetry generators (both in the odd and generic cases) in terms of the Ostrogradski variables. Of course, we expect that the Hamiltonian (3.6) should be transformed into the Ostrogradski one. Indeed, using (4.3)–(4.5) repeatedly we arrive, after straightforward but rather arduous computations (considering two cases: n-odd, even), at the Ostrogradski Hamiltonian (4.7). Similarly, applying Eqs. (4.3)–(4.5), we check that the angular momentum (in both cases (3.12) and (3.30)) transforms under (4.10) into Ostrogradski angular momentum

$$J^{\alpha\beta} = \sum_{k=1}^{n} (Q_k^{\alpha} P_k^{\beta} - Q_k^{\beta} P_k^{\alpha}).$$

As far as the generators C's are concerned (again using (4.3)–(4.5)) we obtain the following expressions:

$$\tilde{C}_i^{-} = \sum_{k=1}^{n} (\cos\omega_i t)^{(k-1)} \tilde{P}_k - \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{j=k}^{n} \sigma_j (\cos\omega_i t)^{(2j-k)} \right) \tilde{Q}_k,$$

$$\tilde{C}_i^{-} = \sum_{k=1}^{n} (\sin\omega_i t)^{(k-1)} \tilde{P}_k - \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{j=k}^{n} \sigma_j (\sin\omega_i t)^{(2j-k)} \right) \tilde{Q}_k, \quad (4.12)$$

in the case of generic frequencies, and

$$\tilde{C}_p = \frac{1}{\omega^p} \sum_{k=1}^{n} \left(\tilde{P}_k (\sin^p \omega t \cos^{2n-1-p} \omega t)^{(k-1)} + (-1)^k \tilde{Q}_k \sum_{j=k}^{n} \sigma_j (\sin^p \omega t \cos^{2n-1-p} \omega t)^{(2j-k)} \right), \quad (4.13)$$

in the odd case; which perfectly agrees with the definitions of the Ostrogradski canonical variables (4.6) and the action of C's on Q (Eqs. (3.9) and (3.16)). Similar reasoning can be done for the remaining two generators D and K in the odd case. Then, they become bilinear forms in the Ostrogradski variables; however the explicit form of coefficients is difficult to simplify and not transparent thus we skip it here.

Algebraic approach to odd case

Since the l-conformal Newton–Hooke algebra is related to the l-conformal Galilei one by the change of Hamiltonian

$$H = \tilde{H} + \omega^2 \tilde{K},$$

where tilde refers to generators of the free theory (which possesses the l-conformal Galilei symmetry); therefore, it would be instructive to construct an alternative Hamiltonian formalism for the PU-model (in the case of odd frequencies) with the help of the one for the free higher derivatives theory.

Denoting by qm,πm, m = 0,..., n − 1 the phase space coordinates of the free theory and adapting the results of Ref. [37] to our conventions we obtain the following form of the generators of the free theory (at time t = 0)

$$\begin{aligned}
\tilde{H} &= \frac{(-1)^{n+1}}{2}\pi_{n-1}^2 - \sum_{m=1}^{n-1}\vec{q}_m\vec{\pi}_{m-1},\\
\tilde{D} &= \sum_{m=0}^{n-1}\left(m - \frac{2n-1}{2}\right)\vec{q}_m\vec{\pi}_m,\\
\tilde{K} &= (-1)^{n+1}\frac{n^2}{2}\vec{q}_{n-1}^{\,2} + \sum_{m=0}^{n-2}(2n-1-m)(m+1)\vec{q}_m\vec{\pi}_{m+1},\\
\tilde{J}^{\alpha\beta} &= \sum_{m=0}^{n-1}\left(q_m^\alpha\pi_m^\beta - q_m^\beta\pi_m^\alpha\right),\\
\tilde{\vec{C}}_m &= (-1)^{m+1}m!\vec{\pi}_m, \quad m = 0,\ldots,n-1,\\
\tilde{\vec{C}}_{2n-1-m} &= (2n-1-m)!\vec{q}_m, \quad m = 0,\ldots,n-1.
\end{aligned} \tag{5.2}$$

Of course, the change of the algebra basis given by (5.1) induces the corresponding one for the coordinates in dual space of the algebra (denoted in the same way); consequently we define the new Hamiltonian as follows

$$\begin{aligned}
H = \tilde{H} + \omega^2\tilde{K} &= \frac{(-1)^{n+1}}{2}\pi_{n-1}^2 - \sum_{m=1}^{n-1}\vec{q}_m\vec{\pi}_{m-1}\\
&+ (-1)^{n+1}\frac{n^2\omega^2}{2}\vec{q}_{n-1}^{\,2} + \sum_{m=0}^{n-2}(2n-1-m)(m+1)\omega^2\vec{q}_m\vec{\pi}_{m+1}.
\end{aligned}$$

We will show that (5.3) is indeed the PU Hamiltonian in qm,πm coordinates and we will find the remaining generators in terms of them. To this end let us define the following transformation

$$\begin{aligned}
\vec{x}_k &= (-1)^k\left(\sum_{m=0}^{n-1}{}'' \frac{\omega^{-m}}{m!\sqrt{|\rho_k|}}\gamma_{km}^+\vec{q}_m + \sum_{m=0}^{n-1}{}' \frac{m!\omega^m\sqrt{|\rho_k|}}{(2k-1)\omega}\beta_{2n-1-m,k}^+\vec{\pi}_m\right),\\
\vec{p}_k &= (-1)^k\left(\sum_{m=0}^{n-1}{}' \frac{\omega^{-m}(2k-1)\omega}{m!\sqrt{|\rho_k|}}\gamma_{k,2n-1-m}^+\vec{q}_m + \sum_{m=0}^{n-1}{}'' m!\omega^m\sqrt{|\rho_k|}\beta_{mk}^+\vec{\pi}_m\right),
\end{aligned}$$

for k = 1,..., n. Using (3.14) and (A.3) we check that (5.4) define a canonical transformation. Moreover, by applying Eqs. (3.14) and (A.2)–(A.4) we check that the PU Hamiltonian (3.6) (with odd frequencies) transforms into (5.3). The remaining generators can be also transformed. First, using (3.14), (A.2), (A.3), (A.5) and (A.6), after troublesome computations, we find that

$$\begin{aligned}
A &= -2\tilde{D},\\
B &= -\tilde{H} + \omega^2\tilde{K},
\end{aligned}$$

and, consequently, we obtain a nice interpretation of A and B. Using Eqs. (5.5), one checks that H, D, K take the form

$$H = \tilde{H} + \omega^2 \tilde{K},$$
$$D = \tilde{D}\cos 2\omega t + \frac{1}{2\omega}\left(\tilde{H} - \omega^2 \tilde{K}\right)\sin 2\omega t,$$
$$K = \frac{1}{2}(1+\cos 2\omega t)\tilde{K} + \frac{1}{2\omega^2}(1-\cos 2\omega t)\tilde{H} + \frac{\sin 2\omega t}{\omega}\tilde{D}.$$

Finally, the angular momentum reads

$$J^{\alpha\beta} = \sum_{m=0}^{n-1}\left(q_m^\alpha \pi_m^\beta - q_m^\beta \pi_m^\alpha\right),$$

i.e., takes the same form as the one for the free theory (according to it commutes with H). The generators Ck are obtained by plugging (5.4) into (3.19) and (3.20), see also (5.17). Summarizing, we expressed all PU symmetry generators in terms of the ones for free theory (and consequently in terms of qm and πm) and we see that the both sets of generators (except Hamiltonian) agree at time t = 0. This result becomes even more evident if we apply the algorithm of constructing integrals of motion for Hamiltonian system with symmetry presented in Ref. [30]. Namely, for the Lie algebra spanned by Xi, i = 1,..., n, [Xi,Xj] = n k=1 ck ijXk , with the adjoint action

$$Ad_g(X_i) = gX_ig^{-1} = \sum_{j=1}^{n} D_i^j(g)X_j,$$

the integrals of motion Xi(ξ,t) corresponding to the generators Xi are of the form

$$X_i(\xi,t) = \sum_{j=1}^{n} D_i^j(e^{tH})\xi_j,$$

where ξ 's are the coordinates of the dual space to the Lie algebra (more precisely, their restriction to the orbits of the coadjoint action in the dual space). Let us apply this approach to our case. One can check that for H, D, K, J αβ Eq. (5.9) gives (5.6) and (5.7). For Cp we have

$$\vec{C}_p = e^{tH}\tilde{\vec{C}}_p e^{-tH} = \sum_{r=0}^{2n-1} a_{pr}(t)\tilde{\vec{C}}_r, \quad p = 0,\ldots,2n-1,$$

where the functions apr satisfy the set of equations

$$\dot{a}_{pr}(t) = (r+1)a_{p,r+1}(t) + (r-2n)\omega^2 a_{p,r-1}(t),$$

with $a_{k,-1} = a_{k,2n} = 0$ and the initial conditions $a_{pr}(0) = \delta_{pr}$. Substituting $a_{pr}(t) = \hat{a}_p$ we obtain

$$\dot{\hat{a}}_{pr}(t) = (r+1)\hat{a}_{p,r+1}(t) + (r-2n)\hat{a}_{p,r-1}(t),$$

Discussion

Let us summarize. In the present paper we focused on the Hamiltonian approaches to the PU model and its symmetries. First, we derived the form of the symmetry generators, in the original Pais and Uhlenbeck approach (for both generic and odd frequencies). We have shown that the resulting algebra is the central extension of the one obtained on the Lagrangian level, i.e., the centrally extended l-conformal Newton–Hooke algebra in the case of odd frequencies and the algebra defined by Eqs. (2.5) and (2.7), in the generic case. Next, we considered the Ostrogradski method of constructing Hamiltonian formalism for theories with higher derivatives. We derived the canonical transformation (Eqs. (4.8)–(4.9)) leading the Ostrogradski Hamiltonian to the one in decoupled oscillators approach.

Let us note that the both approaches, mentioned above, do not distinguish the odd frequencies and in that case do not uncover the richer symmetry. A deeper insight is attained by nothing that for odd frequencies an alternative Hamiltonian formalism can be constructed. It is based on the Hamiltonian formalism for the free higher derivatives theory exhibiting the l-conformal Galilei symmetry. More precisely, we add to the Hamiltonian of the free theory the conformal generator. As a result, we obtain the new Hamiltonian, which turns out to be related, by canonical transformation (5.4), to the PU one. This construction can be better understood from the orbit method point of view, where the construction of dynamical realizations of a given symmetry algebra is related to a choice of one element of the dual space of the algebra as the Hamiltonian (see [30] and the references therein). In our case, both algebras (l-Galilei and l-Newton–Hooke) are isomorphic to each other; only the one generator, corresponding to the Hamiltonian, differ by adding the conformal generator of the free theory. This gives the suitable change in the dual space and consequently the definition (5.3).

The change of the Hamiltonian alters the dynamics, which implies different time dependence of the symmetry generators (which do not commute with H); however, all PU generators should be expressed in terms of the generators of the free theory (for t = 0). This fact was confirmed by applying the method presented in Ref. [30] as well as, directly, by the canonical transformation (5.4) to the decoupled oscillators approach for the PU model.

Turning to possible further developments, let us recall that in the classical case (l = 1 2) the dynamics of harmonic oscillator (on the half-period) is related to the dynamics of free particle by well known Niederer'stransformation [39] (thisfact has also counterpart on the quantum level). It turns out that this relation can be generalized to an arbitrary half-integer l [24] on the Lagrangian level; on the Hamiltonian one, we encounter some difficulties since there is no straightforward transition to the Hamiltonian formalism for a theory with higher derivatives. However, in the recent paper [40] the canonical transformation which relates the Hamiltonian (5.3) to the one for free theory (the first line of (5.2)) has been constructed; it provides a counterpart of classical Niederer's transformation for the Hamiltonian formalism developed in Section 5. Using our results one can obtain similar transformation for both remaining Hamiltonian approaches. We also believe that the results presented here can help in constructing quantum counterpart of the Niederer's transformation for higher l as well as to study of the symmetry of the quantum version of PU oscillator.

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